UNSPLIT ASYMTPTOTIC PRESERVING SCHEMES FOR HYPERBOLIC SYSTEMS WITH RELAXATION

CHRISTINA MAHMOUD AND HÉLÈNE MATHIS

Our study addresses the numerical approximation of hyperbolic systems with relaxation of the form

$$\begin{cases} \partial_t \mathbf{W}^{(1)} + \partial_x \mathbf{f}^{(1)}(\mathbf{W}^{(1)}, \mathbf{W}^{(2)}) = 0, \\ \partial_t \mathbf{W}^{(2)} + \partial_x \mathbf{f}^{(2)}(\mathbf{W}^{(1)}, \mathbf{W}^{(2)}) = \dfrac{1}{\varepsilon}(\mathbf{Q}(\mathbf{W}^{(1)}) - \mathbf{W}^{(2)}), \end{cases} \tag{1}$$

where the parameter $\epsilon > 0$ denotes the relaxation timescale. As ϵ tends to zero, the limit system corresponds to a system of conservation laws

$$\partial_t \mathbf{W}^{(1)} + \partial_x \mathbf{f}^{(1)}(\mathbf{W}^{(1)}, \mathbf{Q}(\mathbf{W}^{(1)})) = 0. \tag{2}$$

The subcharacteristic condition and the compatibility between the entropy of (1) and the entropy of the equilibrium model (2) ensure the stability of the asymptotic.

Standard finite volume approximations of (1) rely on splitting strategy between the convective part and the relaxation source terms. Such strategy may damage the subcharacteristic condition at the discrete level [4]. In the present work, we propose two unsplit finite volume schemes for the approximation of (1), ensuring uniform accuracy in the relaxation parameter ϵ.

The first approach is an adaptation of the implicit staggered scheme [1], originally based on a two-step Roe approach. In our work, we replace the Roe linearization with an asymptotic preserving approximation. We show that, in the appropriate asymptotic limits, the scheme converges to the FORCE scheme [5].

The second scheme corresponds to an approximate Riemann solver adapted from [2], for which we propose a correction to the Godunov step, in order to enforce the scheme to be asymptotic preserving for vanishing ϵ values.

Both schemes enjoy stability properties: preservation of invariant domains, discrete entropy inequality for limiting values of ϵ. Numerical results confirm the good performances of the schemes on both the Jin-Xin [3] and Chaplygin models.

References

[1] F. Bereux and L. Sainsaulieu. A roe-type riemann solver for hyperbolic systems with relaxation based on time-dependent wave decomposition. *Numerische Math.*, 77:143–185, 1997.

[2] C. Berthon, A. Crestetto, and F. Foucher. A well-balanced finite volume scheme for a mixed hyperbolic/parabolic system to model chemotaxis. *J. of Sci. Comp.*, 67:618–643, 2016.

[3] S. Jin and Z. Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Comm. on pure and applied math.*, 48(3):235–276, 1995.

[4] R. B. Pember. Numerical methods for hyperbolic conservation laws with stiff relaxation. I. Spurious solutions. *SIAM J. Appl. Math.*, 53(5):1293–1330, 1993. URL: `https://doi.org/10.1137/0153062`.

[5] E. F. Toro and S. J. Billett. Centred tvd schemes for hyperbolic conservation laws. *IMA Journal of Numerical Analysis*, 20(1):47–79, 2000.

Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, CNRS, Montpellier, France
Email address: `christina.mahmoud@umontpellier.fr`