A general model for shallow water in spherical coordinates

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Abstract

The shallow water equations are a fundamental tool in fluid dynamics, widely used to model the movement of water in situations where the horizontal scales are much larger than the vertical depth. When studying large-scale phenomena like a tsunami propagating through the ocean, the curvature of the Earth must be taken into account, and spherical coordinates must be used in the shallow water equations (SWE).

In this work, we derive a general shallow water system in spherical coordinates. We follow a similar approach as in the derivation of the SWE in Cartesian coordinates, where an average process in the vertical direction is done with the 3D Euler equations. In our case, we do an average process in the normal direction of the sphere, using the 3D Euler equations in spherical coordinates.

Once the general shallow water model is obtained, we compare it both analytically and numerically with the simplified SWE in spherical coordinates that has been used in works such as [1], [2], [3], among many others.

The general SWE derived admits a formulation using conservative variables where the mass of fluid is preserved. This is an improvement with respect to the model used in the literature, since, as it is said in [1], it only preserves the height of the column of water, but not the mass of fluid.

A dimensional analysis shows that the model that is commonly used for many practical applications is in fact a simplification of the general one proposed here. More explicitly, they differ in terms of order H/R, where H is the typical height and R is the radius of the sphere. Therefore, we may expect that if R is much larger than H, both simplified and general models should give similar solutions. Nevertheless, if R is not large enough when compared to H, noticeable differences arise. Several numerical tests have been performed in order to verify this as well as a comparison for the Hugoniot locus of the system.

References

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