Weak PINNs for Hyperbolic Conservation Laws

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Physics-Informed Neural Networks (PINNs) [1] are a machine learning method for approximating solutions to partial differential equations. They are based on minimizing a Monte-Carlo approximation of the L^2 norm of the PDE residual. Due to this, they are considered meshfree methods which are suitable for high-dimensional problems. However, for systems of nonlinear hyperbolic conservation laws, we show that standard PINNs fail at approximating discontinuous solutions using explicit computations.

Our computations imply that it is necessary to use weak (dual) norms of the PDE residual instead, in analogy to moving from the strong form of the PDE to the weak formulation for discontinuous solutions. This approach has been termed "weak PINNs" (wPINNs) [2], wherein an adversarial neural network estimates the dual norm.

Extending wPINNs from [2] we approximate the weak norm computation by solving a family of dual elliptic problems with the PDE residual as right-hand side. This makes learning more stable and effective. Similarly, we introduce and compute an entropy residual, which mimics imposing an entropy inequality on the learned solution. Our modified wPINNs also extend naturally to systems of conservation laws.

For the case of scalar nonlinear hyperbolic conservation laws we also outline how to treat boundary conditions in a weak sense using similar techniques and show the efficacy of this strategy. We verify our technique and extensions in numerical experiments for Burgers equation and the compressible Euler equations, both in one space dimension.

Lastly, we investigate the performance of wPINNs for higher-dimensional problems. We show numerical evidence that even in two dimensions, outside of some basic examples, the performance of wPINNs is significantly worse than in one dimension. We believe this is due to difficulties with quadrature formulas that are used when computing the dual norm. These difficulties stem from the residual being concentrated close to discontinuities in the solution. This may imply that the PINNs approach, despite being meshfree, is not suitable for high-dimensional hyperbolic conservation laws when discontinuities are present.

References

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