# A Lax–Wendroff Theorem for Patankar-Type Schemes

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**ABSTRACT**

For hyperbolic conservation laws, the famous Lax–Wendroff theorem delivers sufficient conditions for the limit of a convergent numerical method to be a weak (entropy) solution. This theorem is a fundamental result, and many investigations have been done to verify its validity for finite difference, finite volume, and finite element schemes, using either explicit or implicit linear time-integration methods.

Recently, the use of modified Patankar (MP) schemes as time-integration methods for the discretization of hyperbolic conservation laws has gained increasing interest. These schemes are unconditionally conservative and positivity-preserving and only require the solution of a linear system. However, MP schemes are by construction nonlinear, which is why the theoretical investigation of these schemes is more involved.

In this talk, we give a brief introduction to Patankar-type methods in the context of hyperbolic conservation laws and present the main ideas of proving a variant of the Lax–Wendroff theorem adding an hypothesis on the total time variation next to the classical total variation (in space) that Lax–Wendroff requires. This extension allows us, for the first time, to establish a theory of their convergence in the context of partial differential equations. The theoretical results are validated through numerical tests for different conservation laws.

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