Discrete Entropy Inequalities, Well-balanced, Robustness and Artificial Viscosity methods

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ABSTRACT

This work studies the stability of numerical schemes for the approximation of the weak solutions of hyperbolic systems of the form:

$$\partial_t w + \partial_x f(w) + A(w)\partial_x w = S(w,x), \qquad x \in \mathbb{R}, \quad t > 0, \quad \omega \in \Omega \subset \mathbb{R}^n. \tag{1}$$

System (1) is assumed to be endowed with an entropy inequality written under the following form:

$$\partial_t \eta(w) + \partial_x G(w) \leq \nabla\eta(w) \cdot S(w,x), \quad \text{in a weak sense,} \tag{2}$$

where $w \mapsto \eta(w)$ is a convex function ($\nabla^2\eta(w) \geq 0$ for all $w \in \Omega$) and $G : \Omega \to \mathbb{R}$ is the entropy flux function satisfying $\nabla G = \nabla\eta \cdot (\nabla f + A)$.

In many particular cases, one may find a general conservative entropy relation in the sense

$$\partial_t \hat{\eta}(w,x) + \partial_x \hat{G}(w,x) \leq 0, \tag{3}$$

where $(w,x) \mapsto \hat{\eta}(w,x)$ is a convex function and $\hat{G} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a generalized entropy flux. In what follows we shall assume that such general pair entropy-entropy flux exists.

Moreover, the previous system has non trivial smooth stationary solutions that satisfy

$$\partial_x f(w) + A(w)\partial_x w = S(w,x), \tag{4}$$

$$\partial_x \hat{G}(w,x) = 0. \tag{5}$$

In the present work, we focus our attention on 3-point finite volume explicit schemes:

$$\begin{aligned} w_i^{n+1} = w_i^n - & \frac{\Delta t}{\Delta x}\left(f_\Delta(w_i^n, w_{i+1}^n) - f_\Delta(w_{i-1}^n, w_i^n)\right) \\ - & \frac{\Delta t}{2\Delta x}\left(A_\Delta^L(w_i^n, w_{i+1}^n)(w_{i+1}^n - w_i^n) + A_\Delta^R(w_{i-1}^n, w_i^n)(w_i^n - w_{i-1}^n)\right) \\ + & \frac{\Delta t}{2}\left(S_\Delta^L(w_i^n, x_i, w_{i+1}^n, x_{i+1}) + S_\Delta^R(w_{i-1}^n, x_{i-1}, w_i^n, x_i)\right), \end{aligned} \tag{6}$$

where f_Δ denotes a consistent numerical flux, $A_\Delta^{L,R}$ are matrices consistent with $A(\cdot)$ and $S_\Delta^{L,R}$ are vectors consistent with $S(\cdot,\cdot)$. Here, w_i^n approximates $w(x,t)$ for all x in a cell $(x_{i-1/2}, x_{i+1/2})$ of size Δx at time t^n. For the sake of simplicity, both Δx and Δt are constant.

Now, some natural questions arise:

- How can we obtain, at the numerical level, an entropy inequality for the scheme?

- Is the numerical method (exactly) well-balanced?
- Do the updated states $(w_i^{n+1})_{i\in\mathbb{Z}}$ be in Ω as soon as $w_i^n \in \Omega$ for all $i \in \mathbb{Z}$?

In the present work, we address these three questions and give a positive answer by adopting an artificial viscosity approach, which is valid for any given 3-point FV first order scheme.

Keywords: finite volume schemes, (exactly) well-balanced methods, robustness, discrete entropy inequalities.

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