Truly multi-dimensional structure-preserving methods on unstructured grids

WASILIJ BARSUKOW (CNRS & Bordeaux University, France)

We present a structure-preserving (mimetic) method on grids of unstructured quads and triangles using cell-node-duals, and show how for fundamental reasons one cannot expect any such methods for pentagons and higher polygonal cells. This is joint work with Raphaël Loubère and Pierre-Henri Maire ([BLM23]).

Previous work on Cartesian methods

Vorticity-preserving methods for linear acoustics

$$\partial_t \mathbf{v} + \nabla p = 0 \qquad\qquad \mathbf{v}\colon \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}^d \tag{1a}$$

$$\partial_t p + \nabla \cdot \mathbf{v} = 0 \qquad\qquad p\colon \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R} \tag{1b}$$

are those which guarantee that a discretization of $\nabla \times \mathbf{v}$ remains stationary. This is only possible if one is able to construct not only a discrete version of $\operatorname{curl}\operatorname{grad} = 0$, but also second derivatives (such as $\operatorname{grad}\operatorname{div}$) that are annihilated by the curl. This is not the case for standard finite differences, but on Cartesian grids suitable finite differences have been identified in e.g. [MR01, JT06, MT09]. In [Bar19] it has been shown that there is in a certain way a unique choice of those.

These finite differences are **truly multi-dimensional**, i.e. they involve also cells which are neighbours only via a node. Also, they have a very special functional form, that becomes intuitive only if one sees the derivatives as being centered on nodes of the grid, and cell-based derivatives being obtained as averages of the expressions centered at adjacent nodes.

Extension to unstructured grids

We are inspired by Riemann solvers originally developed for Lagrangian hydrodynamics, where it is natural to involve a velocity at a node. The consistency condition of Harten-Lax-van Leer [HLL83] imposes equality of numerical fluxes both sides of an interface between cells. We give up this condition and impose that the **sum of all fluxes around a node vanishes**. This is an approach that also appears in residual distribution methods. For it to work out one needs to consider half-edges with possibly different fluxes on each of them (and on each of its sides). We show that when associating a pressure instead of a velocity at each node, a particular choice of such a solver for 2-d linear acoustics admits a discrete analogue of $\operatorname{curl}\operatorname{grad} = 0$ and is **stationarity preserving** and **vorticity preserving**. The nature of this relation is such that grad is defined on the nodes and curl on the cells, or vice versa, extending the Cartesian case. The final scheme is **collocated** due to the usage of averaging operators.

Independently of the precise derivation of the scheme we finally present an argument why for any such node-based solver a discrete version of the $\operatorname{curl}\operatorname{grad} = 0$ identity can only be expected on grids consisting of **polygons with at most 4 edges**.

References

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