Numerical methods for stiff multi-scale hyperbolic problems

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ABSTRACT

Multi-scale problems are omnipresent in environmental and industrial processes, posing a challenge to classical numerical solvers given that the propagation speeds of information span several orders of magnitude. In hyperbolic systems, the absolute fastest wave speed remains finite, but in classical, explicitly integrated numerical schemes, it determines the time step that ensures stability. This means in particular that it may lead to vanishing time increments in the presence of fast processes caused for instance by high pressure or strong magnetic fields. Furthermore, it is well-known that upwind schemes introduce spurious numerical diffusion into the approximate solution [1, 3], a problem that can only be partially mitigated by mesh refinement.

Therefore, a common approach, which is also applied here, is to use implicit or semi-implicit time integrators combined with centred differences for spatial derivatives in implicitly treated systems, in order to obtain scale-independent artificial dissipation and stability under large time steps.

As the evolution of the modelled variables is described by a nonlinear flux function, treating it fully or partially implicitly involves solving nonlinear systems. Depending on the problem, these systems can be large, coupled, ill-posed systems, for which solvers such as the Newton method may converge very slowly or not at all.

To avoid nonlinear implicit systems, we apply Jin-Xin relaxation [2] to the implicitly treated stiff flux terms. This leads to a linear flux structure in the obtained relaxation model. Therein, the nonlinearity of the stiff flux terms is transferred to an algebraic relaxation source term. By splitting away the relaxation source terms, for each variable, a decoupled wave-type equation can be written, yielding a predicted solution at temporarily frozen wave speeds. The final solution is obtained by projecting onto the relaxation equilibrium manifold, taking the relaxation source terms into account, and thus obtaining a prediction-correction scheme for the original hyperbolic multi-scale problem [5, 4]. The properties of the numerical schemes are illustrated at the example of the Euler and mangeto-hydrodynamic (MHD) equations.

Keywords: Hyperbolic multi-scale problem, Jin-Xin relaxation, relaxed schemes, semi-implicit finite volume schemes, prediction-correction scheme

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