On the projected hyperbolic models

M. Parisot
INRIA, Univ. Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, 200 Avenue de la Vieille Tour, 33405 Talence cedex, France (`martin.parisot@inria.fr`)

ABSTRACT

This talk invites the audience to consider a class of models with high order derivatives as hyperbolic models whose solutions are sought within a linear subspace. First we motivate this framework by describing a general strategy to derived approximate models of the water waves problem. Then we delve deeper into this mathematical structure and highlight some mathematical properties. Finally, by preserving this structure at the discrete level, we develop robust and efficient numerical schemes.

The projected hyperbolic models

We focus on hyperbolic models with source term

$$\partial_t U + A\left(U\right)\partial_x U = -\Psi \tag{1}$$

where $U \in \mathbb{R}^{d_U}$, $\Psi \in \mathbb{R}^{d_U}$ and $A \in M_{d_U}\left(\mathbb{R}\right)$ with real eigenvalues. The source term Ψ is not explicitly described, but acts to ensure that the solution remains in a linear subspace $\ker \mathcal{L}$ for a given application $\mathcal{L} : \left(L^2\left(\mathbb{R}\right)\right)^{d_U} \mapsto \left(L^2\left(\mathbb{R}\right)\right)^{d_Q}$. More specifically, the source term Ψ is on the dual space $\ker \mathcal{R} = \left(\ker \mathcal{L}\right)^{\perp}$ such that a Helmholtz decomposition occurs.

Link with the dispersive approximations of the water waves model

Most of the approximate models of the water waves problem can be written under the projected hyperbolic form (1), such as the Korteweg–de Vries, Benjamin-Bona-Mahony and Camassa-Holm models ; the Green-Naghdi and other Boussinesq-type models, and the more complexe dispersive models with several velocities [1, 2]. More precisely, the approximate models can be recovered from a variational formulation of the water waves problem applying a Discontinuous Galerkin vertical discretization of the horizontal velocity which naturally leads to a projected hyperbolic model.

Advantages of the structure for numerical schemes

In a second time, some exemple taken advantage of the projection structure to design robust and efficient numerical schemes will be given. More precisely, by using a splitting between the hyperbolic part and the dispersive source term, the first step reads

$$U^{n*} = U^n - \delta_t A\left(U^n\right)\partial_x U^n \tag{2}$$

that can be solved using classical hyperbolic solver. The second step is nothing more than the Helmholtz decomposition, i.e.

$$\begin{array}{c} U^{n+1} = U^{n*} - \delta_t \Psi^{n+1} \\ \text{with} \qquad \mathcal{L}\left(U^{n+1}\right) = 0 \qquad \text{and} \qquad \mathcal{R}\left(U^{n+1}\right) = 0 \end{array} \tag{3}$$

which highlight the importance to preserve the duality of the operators $\mathcal{L}$ and $\mathcal{R}$ at the discrete level to ensure this step well-posed, and the dissipation of the L^2-norm is ensured. It is worth noting that the Helmholtz decomposition (3) is the most expensive part of the scheme. To reduce this cost, we propose two strategies.

The first is to use high-order scheme. The drawback of Runge-Kutta time schemes is the call of the Helmholtz decomposition (3) at each sub-time steps. Fortunately, following the case of the incompressible flow [3], the dispersive source term can be treated as a time dependent source term, reconstruct from the previous approximations, the Helmholtz decomposition being performed only at the last iteration. This strategy has beed successfully employed for the Green-Naghdi model in [4].

Most of the time and in most areas, the first hyperbolic step (2) gives a good approximation of the result. The second strategy is based on the resolution of the source term Ψ only where and when it is needed. To do so, we first propose a coupling strategy preserving the projection structure [5]. This strategy can also be used to treat discontinuous source terms, like discontinuous bathymetry, and impose practical boundary conditions. In a second time, we propose a a priori criterium selecting the model to minimize the numerical cost.

Keywords: Dispersive equations; Structure preserving scheme; Well-balanced scheme; Entropy-satisfying scheme; High-order scheme;

References

[1] C. Escalante, E. D. Fernández-Nieto, J. Garres-Díaz, T. Morales de Luna, and Y. Penel. Non-hydrostatic layer-averaged approximation of euler system with enhanced dispersion properties. *Computational and Applied Mathematics*, 42(4):177, May 2023.

[2] E. D. Fernández-Nieto, M. Parisot, Y. Penel, and J. Sainte-Marie. A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows. *Communications in Mathematical Sciences*, 16(5):1169–1202, 2018.

[3] J. Guermond, P. Minev, and J. Shen. An overview of projection methods for incompressible flows. *Computer Methods in Applied Mechanics and Engineering*, 195(44):6011 – 6045, 2006.

[4] M. Parisot. Entropy-satisfying scheme for a hierarchy of dispersive reduced models of free surface flow. *International Journal for Numerical Methods in Fluids*, 91(10):509–531, 2019.

[5] M. Parisot. Thick interface coupling technique for weakly dispersive models of waves. *ESAIM: M2AN*, 58(4):1497–1522, 2024.